THE ENRICHED RIEMANN SPHERE AND STABILITY (The second Japanese-Australian Workshop on Real and Complex Singularities)

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In this presentation we will discuss a few suggestive examples, indicating our new approach to Singularity Theory (more details will appear elsewhere).

A general principle which we believe in is that the study of analytic function germs in $n+1$ variables is *Global Analysis* of polynomials in n variables.

This is illustrated here in the case $n=1$. Loosely speaking, the classical Morse Stability Theorem in one variable, properly formulated, is "transplanted" into Algebraic Geometry as theorems on equi-singularities in $\mathbb{C}^2$ (equivalence of singularities); it also suggests a *stronger* definition for "equi-singular deformation".

For example, in contemporary Algebraic Geometry, the following deformations

$$Q(x,y;t) := x^4 - t^2x^2y^2 + y^4, \quad P(x,y;t) := x^3 - y^4 - 3t^2xy^k,\ k \geq 3, \tag{0.1}$$

are equi-singular, because their zero sets are topologically trivial (Milnor μ-constant).

However, Q is *not* equi-singular from our point of view. The hypothesis of our Equi-singularity Theorem is not satisfied. The associated family of polynomials $x^4 - t^2x^2 + 1$ is not Morse stable ($x=0$ splits into three critical points when $t \neq 0$).

On the other hand, the Pham family P *is equi-singular in our sense* . (Even though the "polar" $\partial P/\partial x$ splits into distinct factors $x \pm ty^d$ if $k = 2d$.) The associated family $x^3 - 1$, being independent of t, is obviously Morse stable. By our Equi-singularity Theorem, P itself, *not merely* the zero set, admits a trivialization.

1. Morse Stability

When does a given family $F(x,y;t)$, like Q, P above, admit a trivialization, and of what kind? This is answered in our Equi-singularity Theorem, modelled on the classical Morse Theorem. The Morse Stability Theorem over $\mathbb{F}$ is also geometrized.

Definition 1.1. Given $p_t(x) := a_0(t)x^n + \cdots + a_n(t) \in \mathbb{K}\{t\}[x]$, as a deformation of $p_0(x)$, $a_0(t) \neq 0$, $t \in I_{\mathbb{K}}$, where $\mathbb{K} := \mathbb{C}$ or $\mathbb{R}$. A critical point $c \in \mathbb{K}$ of $p_0(x)$ is *stable* if it admits a *continuous* deformation $c_t \in \mathbb{K}$, a critical point of $p_t(x)$, with $m_{crit}(c_t) = m_{crit}(c)$. (See Example (1.2).)

The deformation $\{p_t\}$ is *Morse stable* if the following hold.

(1) Every critical point $c \in \mathbb{K}$ of $p_0(x)$ is stable ;

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(2) If $p_0(c) = p_0(c')$, c, c' critical points of $p_0(z)$, then $p_t(c_t) = p_t(c'_t)$, $t \in I_{\mathbb{K}}$;
(3) If $p_0(c) = p'_0(c) = 0$, *i.e.*, c is a multiple root of $p_0(z)$, then $p_t(c_t) = 0$, $t \in I_{\mathbb{K}}$.

Conditions (1), (2) come from Morse Theory; (3) is new, needed for Algebraic Geometry. A version of the classical Morse Stability Theorem is the following.

The Morse Theorem. *Suppose $\{p_t(x)\}$ is Morse stable. There exist t-level preserving homeomorphisms $\mathcal{D} : \mathbb{K} \times I_{\mathbb{K}} \to \mathbb{K} \times I_{\mathbb{K}}$, and $\delta : \mathbb{K} \times I_{\mathbb{K}} \to \mathbb{K} \times I_{\mathbb{K}}$,*

$$\mathcal{D} : (x,t) \mapsto (D_t(x), t); \quad \delta : (v,t) \mapsto (d_t(v), t), \quad d_t(0) = 0, \tag{1.1}$$

where $D_0(x) = x$, $d_0(v) = v$, such that $p_t(D_t(x)) = d_t(p_0(x))$, and c is a critical point of p_0 iff $D_t(c)$ is one of p_t. (Note that $p_0(a) = 0$ iff $p_t(D_t(a)) = 0$.)

$$I_{\mathbb{R}} := \{t \in \mathbb{R} \mid |t| < \epsilon\}, \quad I_{\mathbb{C}} := \{t \in \mathbb{C} \mid |t| < \epsilon\}, \quad I_{\mathbb{F}} := \{t \in \mathbb{D} \mid |t| < \epsilon\}, 1 >> \epsilon > 0. \tag{1.2}$$

Here $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or the Newton-Puiseux field $\mathbb{F}$. The "disk" $\mathbb{D} \subset \mathbb{F}$ is described in the next section.

Example 1.2. Take $\mathbb{K} = \mathbb{R}$. For $p_t(x) = x^2(x^2 + t^2) \in \mathbb{R}[x]$, 0 is a critical point of p_0 which splits into 3 critical points in $\mathbb{C}$, one remains in $\mathbb{R}$. Thus 0 admits a *unique continuous* deformation $c_t \equiv 0$ *in* $\mathbb{R}$. But $m_{crit}(c_t)$ is not constant, 0 is *unstable.*

2. The Enriched Riemann Sphere

The Riemann sphere $\mathbb{C}P^1$ is "enriched" to $\mathbb{C}P^1_*$ with "infinitesimals", which are irreducible curve germs; and $\mathbb{C}$ enriched to $\mathbb{C}_*$. The Newton-Puiseux field $\mathbb{F}$ provides coordinate systems, in terms of which several structures are defined.

The Cauchy Integral Theorem, Taylor expansions, critical points, stability, *etc.*, are generalized to $\mathbb{F}$; and so is the classical Morse Stability Theorem.

Take a holomorphic map germ $\mathcal{A} : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $\mathcal{A}(z) \neq 0$ if $z \neq 0$. The image *set germ*, $Im(\mathcal{A})$, or the *geometric locus* of $\mathcal{A}$, has a well-defined tangent line, $T(\mathcal{A})$, at 0. We call $Im(\mathcal{A})$ an *infinitesimal at* $T(\mathcal{A}) \in \mathbb{C}P^1$. The set of infinitesimals is denoted by $\mathbb{C}P^1_*$.

The geometric locus of $z \mapsto (az, bz)$ is identified with $[a : b] \in \mathbb{C}P^1$; hence $\mathbb{C}P^1 \subset \mathbb{C}P^1_*$.

For example, the curve germ $x^2 - y^3 = 0$, as the geometric locus of $z \mapsto (z^3, z^2)$, is an infinitesimal at $[0 : 1]$. It is "closer" to $[0 : 1]$ than any $[a : 1]$, $a \neq 0$.

As in Projective Geometry, $\mathbb{C}P^1_*$ is a union $\mathbb{C}P^1_* = \mathbb{C}_* \cup \mathbb{C}'_*$, where

$$\mathbb{C}_* := \{Im(\mathcal{A}) \mid T(\mathcal{A}) \neq [1:0]\}, \quad \mathbb{C}'_* := \{Im(\mathcal{A}) \mid T(\mathcal{A}) \neq [0:1]\}.$$

The classical Newton-Puiseux Theorem asserts that the field $\mathbb{F}$ of convergent fractional power series in an indeterminate y is algebraically closed.

Recall that a non-zero element of $\mathbb{F}$ is a (finite or infinite) convergent series

$$\alpha : \alpha(y) = a_0 y^{n_0/N} + a_1 y^{n_1/N} + \cdots, \quad a_i \neq 0, \quad n_0 < n_1 < \cdots, \tag{2.1}$$

where $n_i \in \mathbb{Z}$, $N \in \mathbb{Z}^+$, $a_i \in \mathbb{C}$. The *order* of α is $O_y(\alpha) := n_0/N$; $O_y(0) := +\infty$.

We can assume $GCD(N, n_0, n_1, \ldots) = 1$. The *Puiseux multiplicity* of α is $m_{puis}(\alpha) := N$. The *conjugates* of α are $\alpha^{(k)}_{conj}(y) := \sum a_i \theta^{kn_i} y^{n_i/N}$, $0 \le k \le N-1$, where $\theta := e^{2\pi i/N}$.

The following $\mathbb{D}$ is an integral domain with quotient field $\mathbb{F}$ and maximal ideal $\mathbb{M}$,

$$\mathbb{D} := \{\alpha \in \mathbb{F} \mid O_y(\alpha) \ge 0\},\ \ \mathbb{M} := \{\alpha \mid O_y(\alpha) > 0\},\ \ \mathbb{M}_1 := \{\alpha \mid O_y(\alpha) \ge 1\};$$

$\mathbb{M}_1$ is an ideal. Define $|\alpha| := \sum 2^{-n_i/N}|a_i|(1+|a_i|)^{-1}$, $d(\alpha,\beta) := |\alpha - \beta|$ is a *metric* on $\mathbb{D}$.

Thus, $\lim_{m\to\infty} \sum a_i(m) y^{n_i/N} = 0$ *iff* each $a_i(m) \to 0$, the point-wise convergence.

Given $\alpha \in \mathbb{M}_1$, let $\mathcal{A}(z) := (\alpha(z^N), z^N)$, $N := m_{puis}(\alpha)$. We define $\alpha_* := \pi_*(\alpha) := Im(\mathcal{A})$, and use $\pi_* : \mathbb{M}_1 \to \mathbb{C}_*$, a many-to-one surjective mapping, as a *coordinate system* on $\mathbb{C}_*$.

A *coordinate system* on $\mathbb{C}'_*$ is $\pi'_* : \mathbb{M}_1 \to \mathbb{C}'_*$, $\alpha_* := \pi'_*(\alpha) := Im(\mathcal{A})$, $\mathcal{A}(z) := (z^N, \alpha(z^N))$.

Let $\mathbb{C}_*$ (resp. $\mathbb{C}'_*$) be furnished with the quotient topology of π_* (resp. π'_*). As for the transition function in the overlap $\mathbb{C}_* \cap \mathbb{C}'_*$, take $x = \alpha(y)$, $n_0/N = 1$, we then "solve y in terms of x", obtaining $y = \beta(x) := b_0 x + b_1 x^{n'_1/N'} + \cdots$, $a_0 b_0 = 1$, each b_i is a *polynomial* in finitely many of $(\sqrt[N]{a_0})^{-1}$, a_1/a_0, $a_2/a_0, \ldots$. Hence the topologies coincide in $\mathbb{C}_* \cap \mathbb{C}'_*$.

The quotient topology on $\mathbb{C}P^1_*$ *is well-defined.*

Next, let $X, Y \subset \mathbb{R}^n$ be germs of sub-analytic sets at 0, $X \cap Y = \{0\}$, $X \neq \{0\} \neq Y$. The *contact order* $O(X,Y)$ is, *by definition*, the smallest number L (the Lojasiewicz exponent) such that $d(x,y) \ge a\|(x,y)\|^L$, where $x \in X$, $y \in Y$, $\|x\| = \|y\|$, $a > 0$ a constant.

Hence $O(\alpha_*, \beta_*)$ is well-defined, $O(\alpha_*, \alpha_*) := \infty$. (Example: for α, $\beta \in \mathbb{M}_1$, $O(\pi_*(\alpha), \pi_*(\beta)) = \max_{k,j}\{O_y(\alpha^{(k)}_{conj} - \beta^{(j)}_{conj})\}$.) *This is the contact order structure on* $\mathbb{C}P^1_*$.

The *enriched Riemann Sphere* is $\mathbb{C}P^1_*$ furnished with the above structures; $\mathbb{C}_*$ is the *enriched complex plane.*

3. Equi-singularity Theorem

Given $f(x,y) \in \mathbb{C}\{x,y\}$, *mini-regular* in x of order m, *i.e.*,

$$f(x,y) = H_m(x,y) + H_{m+1}(x,y) + \cdots,\ H_m(1,0) \neq 0,\ H_i(x,y)\ i\text{-form}.$$

Take a deformation $F(x,y;t) = \sum_{i+j\ge m} c_{ij}(t)x^i y^j \in \mathbb{C}\{x,y,t\}$, $F(x,y;0) = f(x,y)$.

Define $\phi_t(\xi) := F(\xi, y; t)$, $\xi \in \mathbb{M}_1$, $\Phi := \{\phi_t\}$, $t \in I_{\mathbb{C}}$.

The Equi-singularity Theorem. *Suppose* Φ *is Morse stable. There exists a map germ*

$$H : (\mathbb{C}^2 \times I_{\mathbb{C}}, 0 \times I_{\mathbb{C}}) \to (\mathbb{C}^2 \times I_{\mathbb{C}}, 0 \times I_{\mathbb{C}}),\quad ((x,y),t) \mapsto (\eta_t(x,y),t), \tag{3.1}$$

which is a homeomorphism, real bi-analytic outside $\{0\} \times I_{\mathbb{C}}$*, such that*

(1) $F(\eta_t(x,y);t) = f(x,y)$, $t \in I_{\mathbb{C}}$, *i.e.*, $F(x,y;t)$ *is "trivialized" by* H*;*

(2) $H_* : \mathbb{C}P^1_* \times I_{\mathbb{C}} \to \mathbb{C}P^1_* \times I_{\mathbb{C}}$, $(\alpha_*, t) \mapsto (\eta_t(\alpha_*), t)$, is a homeomorphism, where $\eta_t(\alpha_*)$ as a set germ is a point of $\mathbb{C}P^1_*$ (we do not claim that if $\mathcal{A}$ is holomorphic then so is $\eta_t \circ \mathcal{A}$);

(3) The contact order is preserved: $O(\alpha_*, \beta_*) = O(\eta_t(\alpha_*), \eta_t((\beta_*))$;

(4) The Puiseux pairs is preserved: $\chi_{puis}(\eta_t(\alpha_*)) = \chi_{puis}(\alpha_*)$;

(5) There exists a constant $\varepsilon > 0$, $\varepsilon \leq \|\eta_t(x, y)\| / \|(x, y)\| \leq 1/\varepsilon$, $t \in I_{\mathbb{C}}$;

(6) If $\mathcal{R} : (\mathbb{R}, 0) \to (\mathbb{C}^2, 0)$ is (real-)analytic then so is $\eta_t \circ \mathcal{R}$, i.e., η_t is arc-analytic.

The proof of the Equi-singularity Theorem above, uses a vector field $\vec{\mathcal{F}}(x, y, t)$, $(x, y, t) \in U \times I_{\mathbb{C}}$.

There exists $\gamma(y) := \gamma_\phi(y) + \cdots$, $F_x(\gamma(y), y; 0) = 0$; *i.e.*, $F_x(x, y; 0)$ vanishes on the curve germ $\Delta := \pi_*(\gamma)$ which is customarily called a "polar" of $F(x, y; 0)$.

Let Δ_t denote the image of Δ at time t in the flow. Note that the above *does not imply* that Δ_t is a polar of $F(x, y; t)$.

The set $\mathcal{P}(\Gamma) := \{\Delta \in \mathbb{C}_* | O(\Delta, \Gamma) > O(\gamma_\phi)\}$ contains at least one polar of $F(x, y; 0)$. Hence we call $\mathcal{P}(\Gamma)$ a *blurred polar*, and Γ its *canonical representative.*

As we shall prove, the flow *preserves* the contact order, hence induces a bijection between $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma_t)$. *The flow only carries one blurred polar to another.*

The Pham family $P(x, y; t)$ in (0.1), $k = 2d$, has *two* polars when $t \neq 0$, but only *one* blurred polar. The blurred polar *is invariant* under the flow; the polars are not. Nevertheless this suffices for showing the triviality of the Pham family.

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